



Context for Models of Concurrency

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Abstract

Many categories have been used to model concurrency. Using any of these, the challenge is to reduce a given model to a smaller representation which nevertheless preserves the relevant computer-scientific information. That is, one wants to replace a given model with a simpler model with the same directed homotopy-type. Unfortunately, the obvious definition of directed homotopy equivalence is too coarse. This paper introduces the notion of *context* to refine this definition.

Keywords: models for concurrency, po-space (pospace), directed homotopy (dihomotopy), context, fundamental category, model category, pushout, local po-space (local pospace)

1 Introduction

Various topological models are being used for studying concurrency. Among them are precubical complexes [7], d-spaces [9,10], local po-spaces [4,8], and FLOW [6]. For a given concurrent system, each of these categories provides a model which captures the relevant computer-scientific properties of the system.

These categories are large in two senses. They are large ‘locally’ in that a given model contains many paths which correspond to executions which are essentially equivalent. They are also large ‘globally’ in that a given concurrent system has a large number of models within the category. The size of these categories is a strength in terms of their descriptive power. However, to aid in calculations one would like to reduce these models to a smaller, possibly even discrete, representation.

A major goal of current research in this area is to introduce equivalences to obtain such smaller representations, which nevertheless still retain the relevant computer-scientific properties.

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On the local front progress has been made in reducing the path space of a given model using directed homotopies of paths and the fundamental category [9]. One global approach is to pass to the component category [5,13]. In this paper we introduce another global approach, which is perhaps more geometric and which is compatible with the model categorical approach of [1].

In the classical (undirected) topological case, the solution to this ‘global’ problem is well-understood. The equivalent spaces are the (weak) homotopy equivalent ones. So for example, all of the contractible spaces (those homotopy equivalent to a point) are equivalent.

In the directed case there is a similar notion of directed homotopy equivalence (abbreviated to *dihomotopy equivalence*, which will be defined in the next section). However this notion is too coarse.

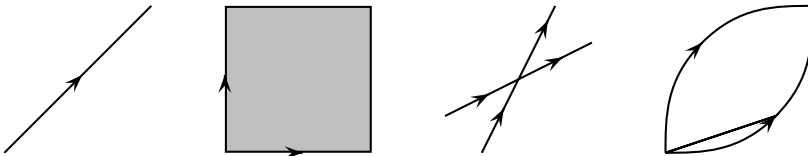


Fig. 1. \vec{I} , $\vec{I} \times \vec{I}$, \vec{X} , and \vec{O}

Example 1.1 Let \vec{I} be the unit interval $[0, 1]$ with a direction given by the usual ordering of the real numbers. Let $\vec{I} \times \vec{I}$ be $[0, 1] \times [0, 1]$ with the ordering $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Let \vec{X} be the space in Figure 1 given by attaching two copies of \vec{I} at their midpoints. Then as will be shown explicitly in Example 2.7, \vec{I} , $\vec{I} \times \vec{I}$ and \vec{X} are all dihomotopy equivalent to a point. However \vec{I} models an execution with one initial state and one final state while \vec{X} models an execution with two initial states and two final states.

Clearly a stronger notion of equivalence is needed. Since \vec{I} and $\vec{I} \times \vec{I}$ both have one initial state and one final state and all execution paths seem to be essentially equivalent it seems natural that we should look for a definition of equivalence under which these are equivalent. However even this ‘equivalence’ has a pitfall.

For a notion of equivalence to be practical it should continue to hold under certain ‘pastings’. Our philosophy is the following. If we make the same addition to equivalent models we should still have equivalent models.

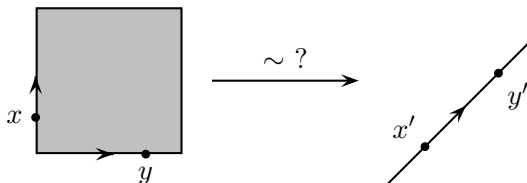
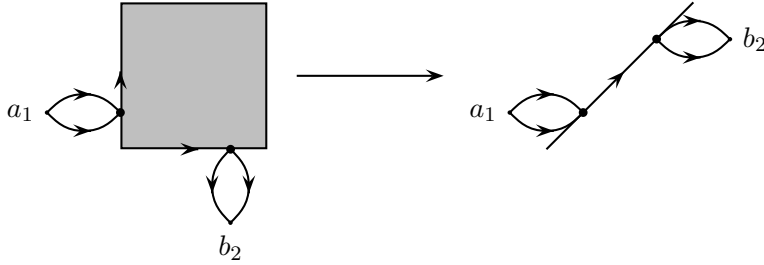


Fig. 2. A hypothetical equivalence

Example 1.2 Assume we have an equivalence $\vec{I} \times \vec{I} \rightarrow \vec{I}$ as in Figure 2. Consider the following pasting on $\vec{I} \times \vec{I}$. Let \vec{O} be the space in Figure 1 constructed by

Fig. 3. A map $B \rightarrow C$ which should not be an equivalence

attaching two copies of \vec{I} at their initial points and at the final points.³ Let \vec{O}_1 and \vec{O}_2 be two copies of \vec{O} . For $i = 1, 2$ let $a_i, b_i \in \vec{O}_i$ denote the initial and final points of \vec{O}_i . Now choose two points $x, y \in \vec{I} \times \vec{I}$ such that neither $x \leq y$ nor $y \leq x$. Let $x', y' \in \vec{I}$ be the images of x and y under the assumed equivalence (Figure 2). Then either $x' \leq y'$ or $y' \leq x'$, since \vec{I} is totally ordered.

If $x' \leq y'$ then identify b_1 and x and identify a_2 and y . Call this space B and denote C the space obtained by collapsing $\vec{I} \times \vec{I} \subset B$ to \vec{I} using the given equivalence (Figure 3). Then there is an execution path from a_1 to b_2 in C but not in B . So the concurrent systems modeled by B and C are not equivalent. A similar construction is possible if $y' \leq x'$. Thus from this point of view $\vec{I} \times \vec{I}$ and \vec{I} should not be equivalent.

This gives a good indication of the current state of affairs for determining a global notion equivalence. We don't even know whether or not $\vec{I} \times \vec{I}$ and \vec{I} should be equivalent.

In this paper we introduce the idea of *context*. Whether or not \vec{I} and $\vec{I} \times \vec{I}$ are equivalent depends on the context. If we permit pastings as in Example 1.2, then they are not equivalent. However if we only permit pastings to the initial and final points of \vec{I} and $\vec{I} \times \vec{I}$ then they are equivalent. From the computer-scientific point of view this can be interpreted as follows. We cannot expect equivalent concurrent systems to still be equivalent after arbitrary (but equal) changes. However, if equal additions are made in a suitably modular way, then the resulting systems should still be equivalent.

It should be noted that in the examples in this paper the context is chosen 'by hand'. The problem of choosing the context is related to the components of the fundamental category [5] and to the universal discovering space [3]. A procedure for choosing the context is a subject for future research.

Acknowledgments. The author would like to thank the referee for simplifying the proof of Proposition 3.5.

2 Context for directed homotopy equivalences

In this section we make precise the intuitive ideas presented in the introduction.

³ This is M. Grandis' *ordered circle* $\uparrow O^1$ [9, Section 1.2].

Definition 2.1 • A *partial order* on a topological space U is a reflexive, transitive, anti-symmetric relation \leq . If U has a partial order \leq which is a closed subset of $U \times U$ under the product topology, then call U a *po-space* or *pospace*.

- A *dimap* $f : (U_1, \leq_1) \rightarrow (U_2, \leq_2)$ is a continuous map $f : U_1 \rightarrow U_2$ such that $x \leq_1 y$ implies that $f(x) \leq_2 f(y)$.
- A product of pospaces (U_1, \leq_1) and (U_2, \leq_2) is a pospace whose underlying topological space is $U_1 \times U_2$ and whose order relation is given by $(x, y) \leq (x', y')$ if and only if $x \leq_1 x'$ and $y \leq_2 y'$.
- A subspace A of a pospace U inherits a pospace structure under the definition $x \leq_A y$ if and only if $x \leq_U y$. This is called a *sub-pospace*.

Definition 2.2 Let **Pospace** be the category whose objects are pospaces and whose morphisms are dimaps.

For the sake of simplicity we will work with pospaces but one should be able to easily extend or adapt the constructions presented here for other models of concurrency.

Let $\vec{I} = ([0, 1], \leq)$ where \leq is the usual ordering of \mathbb{R} . This is a pospace. A *dipath* in a pospace B is a dimap $\vec{I} \rightarrow B$.

Definition 2.3 • Given dimaps $f, g : B \rightarrow C \in \mathbf{Pospace}$, $\phi : B \times \vec{I} \rightarrow C \in \mathbf{Pospace}$ is a *dihomotopy*⁴ from f to g if $\phi|_{B \times \{0\}} = f$ and $\phi|_{B \times \{1\}} = g$. In this case write $\phi : f \rightarrow g$.

- Write $f \simeq g$ if there is a chain of dihomotopies $f \rightarrow f_1 \leftarrow f_2 \rightarrow \dots \leftarrow f_n \rightarrow g$. This is an equivalence relation.
- A dimap $f : B \rightarrow C$ is a *dihomotopy equivalence* if there is a dimap $g : C \rightarrow B$ such that $g \circ f \simeq \text{Id}_B$ and $f \circ g \simeq \text{Id}_C$. In this case write $B \simeq C$.

Our explicit dihomotopies will often be of the following form.

Definition 2.4 Assume that C is a pospace whose underlying topological spaces is a subspace of \mathbb{R}^n for some n . Assume $f, g : B \rightarrow C$ are two dimaps. Let the *linear interpolation* between f and g be the map $H : B \times \vec{I} \rightarrow \mathbb{R}^n$ given by $H(b, t) = (1 - t)f(b) + tg(b)$.

Remark 2.5 Note that there is no guarantee that the image of such a map is in C . However one can check that it is for the cases we will consider.

Lemma 2.6 Assume that C is a pospace whose underlying topological spaces is a subspace of \mathbb{R}^n for some n . If $f, g : B \rightarrow C$ are dimaps such that for all $b \in B$, $f(b) \leq g(b)$ then if the image of the linear interpolation H between f and g is in C then H is a dihomotopy from f to g .

Proof. That H restricts to f and g follows from the definition of linear interpolation. It remains to check that H is a dimap.

⁴ This is the notion of dihomotopy in [9] which is stronger than the notion of dihomotopy in [4] (which uses $I = [0, 1]$ with the trivial ordering $x \leq_I y \Leftrightarrow x = y$, instead of \vec{I}).

Let $b \leq_B b'$ and $t \leq_{\vec{T}} t'$. Then

$$\begin{aligned} H(b, t) &= f(b) + t(g(b) - f(b)) \\ &\leq f(b) + t'(g(b) - f(b)) \\ &= (1 - t')f(b) + t'g(b) \\ &\leq (1 - t')f(b') + t'g(b') \\ &= H(b', t') \end{aligned}$$

□

Example 2.7 If \vec{T}_1 and \vec{T}_2 are two copies of \vec{T} , then let $\vec{X} = (\vec{T}_1 \amalg \vec{T}_2) / \sim$ where $(\frac{1}{2})_1 \sim (\frac{1}{2})_2$ (see Figure 1). We will show that under Definition 2.3, \vec{T} , $\vec{T} \times \vec{T}$, and \vec{X} are dihomotopy equivalent to a point. Let $f : \vec{T} \rightarrow *$, $g : * \rightarrow \vec{T}$ be the constant map and the inclusion of the point to $1 \in \vec{T}$. Then $f \circ g = \text{Id}_*$ and it remains to show that $\text{Id}_{\vec{T}} \simeq g \circ f$. Let $H : \vec{T} \times \vec{T} \rightarrow \vec{T}$ be the linear interpolation between $\text{Id}_{\vec{T}}$ and $g \circ f$. That is,

$$\begin{aligned} H(x, t) &= (1 - t)x + t \\ &= x + t(1 - x) \end{aligned}$$

Then H is a dimap and is the desired homotopy $\text{Id}_{\vec{T}} \rightarrow g \circ f$.

In exactly the same way one can show that the constant map $f : \vec{T} \times \vec{T} \rightarrow *$ is a dihomotopy equivalence with $g : * \rightarrow \vec{T} \times \vec{T}$ given by $g(*) = (1, 1)$.

To show that the constant map $f : \vec{X} \rightarrow *$ is a dihomotopy equivalence with $g(*) = (\frac{1}{2})_1 = (\frac{1}{2})_2$ is slightly more complicated. Again $f \circ g = \text{Id}_*$. To show $\text{Id}_{\vec{X}} \simeq g \circ f$ we will construct a chain of dihomotopies $\text{Id}_{\vec{X}} \xrightarrow{H_1} h \xleftarrow{H_2} g \circ f$. Let h be the map that collapses the lower two line segments of \vec{X} . That is, let $h : \vec{X} \rightarrow \vec{X}$ be given by

$$x \mapsto \begin{cases} \frac{1}{2} & \text{if } x < \frac{1}{2} \\ x & \text{otherwise} \end{cases}$$

Let H_1 be the linear interpolation between $\text{Id}_{\vec{X}}$ and h and let H_2 be the linear interpolation between $g \circ f$ and h . Then H_1 and H_2 are dimaps and are the desired dihomotopies.

We will show that in the right *context* it is no longer true that \vec{T} , $\vec{T} \times \vec{T}$, and \vec{X} are dihomotopy equivalent to a point.

Definition 2.8 Let the *context* be an object $A \in \mathbf{Pospace}$. Instead of working in the category $\mathbf{Pospace}$ we will work in the category $\mathbf{A} \downarrow \mathbf{Pospace}$ of pospaces under A . The objects of $\mathbf{A} \downarrow \mathbf{Pospace}$ are dimaps $A \xrightarrow{\iota_B} B$ where $B \in \text{Ob } \mathbf{Pospace}$. The morphisms in $\mathbf{A} \downarrow \mathbf{Pospace}$ are dimaps

$$\begin{array}{ccc} & A & \\ \iota_B \swarrow & & \searrow \iota_C \\ B & f & C \end{array}$$

such that $f \circ \iota_B = \iota_C$.

Example 2.9 For example if $A = S^0 = \{a, b\}$ then $B \in \mathbf{Ob} \mathbf{A} \downarrow \mathbf{Pospace}$ is a pospace with two marked points. An important example is \overrightarrow{I} with $\iota_{\overrightarrow{I}}(a) = 0$ and $\iota_{\overrightarrow{I}}(b) = 1$.

Definition 2.10 • Given dimaps $f, g : B \rightarrow C \in \mathbf{A} \downarrow \mathbf{Pospace}$, ϕ is a *dihomotopy* from f to g if $\phi : B \times \overrightarrow{I} \rightarrow C \in \mathbf{Pospace}$, $\phi|_{B \times \{0\}} = f$, $\phi|_{B \times \{1\}} = g$, and for all $a \in A$, $\phi(\iota_B(a), t) = \iota_C(a)$. In this case write $\phi : f \rightarrow g$.

- Write $f \simeq g$ if there is a chain of dihomotopies $f \rightarrow f_1 \leftarrow f_2 \rightarrow \dots \leftarrow f_n \rightarrow g$. This is an equivalence relation.
- A dimap $f : B \rightarrow C$ is a *dihomotopy equivalence* if there is a dimap $g : C \rightarrow B$ such that $g \circ f \simeq \text{Id}_B$ and $f \circ g \simeq \text{Id}_C$. In this case write $B \simeq C$.

We can think of this as *dihomotopy rel A*. In case the context A is one point or two points we get pointed and bipointed dihomotopies. However we will see that this notion is useful for more general contexts.

Example 2.11 Let us return to the example above. In the context of its end points \overrightarrow{I} is no longer dihomotopic to a point. There is a dimap

$$\begin{array}{ccc} & S^0 & \\ \iota_{\overrightarrow{I}} \swarrow & & \searrow \iota_* \\ \overrightarrow{I} & f & * \end{array}$$

making the diagram commute, but there is no map $g : * \rightarrow \overrightarrow{I}$ making the diagram commute.

Example 2.12 In the context of $S^0 = \{a, b\}$ let $\iota_{\overrightarrow{I}}(a) = 0$, $\iota_{\overrightarrow{I}}(b) = 1$, $\iota_{\overrightarrow{I} \times \overrightarrow{I}}(a) = (0, 0)$, and $\iota_{\overrightarrow{I} \times \overrightarrow{I}}(b) = (1, 1)$. We claim that in this context \overrightarrow{I} and $\overrightarrow{I} \times \overrightarrow{I}$ are dihomotopy equivalent. Let $f : \overrightarrow{I} \times \overrightarrow{I} \rightarrow \overrightarrow{I}$ and $g : \overrightarrow{I} \rightarrow \overrightarrow{I} \times \overrightarrow{I}$ be given by $f(x, y) = \max(x, y)$ and $g(x) = (x, x)$. Then f and g are both dimaps, $f \circ g = \text{Id}_{\overrightarrow{I}}$ and $g \circ f(x, y) = (\max(x, y), \max(x, y))$. It remains to construct a dihomotopy rel S^0 from $\text{Id}_{\overrightarrow{I} \times \overrightarrow{I}}$ to $g \circ f$.

Let ϕ be the linear interpolation (see Definition 2.4) of $\text{Id}_{\overrightarrow{I} \times \overrightarrow{I}}$ and $g \circ f$. That is,

$$\begin{aligned} \phi(x, y, t) &= (1 - t)(x, y) + t(\max(x, y), \max(x, y)) \\ &= (x + t(\max(x, y) - x), y + t(\max(x, y) - y)). \end{aligned}$$

Then ϕ is the desired dihomotopy rel $\{a, b\}$.

Hence $\overrightarrow{I} \times \overrightarrow{I}$ and \overrightarrow{I} are dihomotopy equivalent in the given context.

3 Context and the fundamental category

We will now introduce some definitions and prove some lemmas that will allow us to relate dihomotopy $\text{rel } A$ to the *fundamental category*. Furthermore it will enable us to quickly see that certain spaces are not dihomotopy equivalent in a given context.

Definition 3.1 Let $B \in \mathbf{Pospace}$ and let $x, y \in B$.

- A *dipath* is a dimap $\gamma : \vec{I} \rightarrow B$.
- Let $\gamma_1, \gamma_2 : \vec{I} \rightarrow B$ be dipaths such that $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1(1) = \gamma_2(2) = y$. Then γ_1 and γ_2 are dihomotopic if they are dihomotopy equivalent with respect to their endpoints. That is, $\gamma_1 \simeq \gamma_2$ in $\mathbf{S}^0 \downarrow \mathbf{Pospace}$ where $\iota_{\vec{I}}(a) = 0$, $\iota_{\vec{I}}(b) = 1$, $\iota_B(a) = x$, and $\iota_B(b) = y$. In this case write $\gamma_1 \simeq \gamma_2$.
- Let $\vec{\pi}_1(B)(x, y)$ be the set of dihomotopy classes of dipaths from x to y . The *fundamental category* of B is the category $\vec{\pi}_1(B)$ whose objects are the points of B and whose morphisms between x and y are the elements of $\vec{\pi}_1(B)(x, y)$.⁵

Lemma 3.2 Given dihomotopic dipaths $\gamma \simeq \gamma' : \vec{I} \rightarrow B$ and a dimap $f : B \rightarrow C$, then $f \circ \gamma \simeq f \circ \gamma'$ are dihomotopic dipaths.

Proof. Since $\gamma \simeq \gamma'$ there is a chain of dihomotopies $\gamma \xrightarrow{H_1} \gamma_1 \xleftarrow{H_2} \gamma_2 \xrightarrow{H_3} \dots \xleftarrow{H_n} \gamma_n \xrightarrow{H_{n+1}} \gamma'$. Then $f \circ \gamma \xrightarrow{f \circ H_1} f \circ \gamma_1 \xleftarrow{f \circ H_2} f \circ \gamma_2 \xrightarrow{f \circ H_3} \dots \xleftarrow{f \circ H_n} f \circ \gamma_n \xrightarrow{f \circ H_{n+1}} f \circ \gamma'$ is a chain of dihomotopies from $f \circ \gamma$ to $f \circ \gamma'$. \square

Corollary 3.3 For a dimap $f : B \rightarrow C$ and $x, y \in B$ there is an induced map $\vec{\pi}_1(f) : \vec{\pi}_1(B)(x, y) \rightarrow \vec{\pi}_1(C)(f(x), f(y))$ mapping $[\gamma] \mapsto [f \circ \gamma]$. That is, a dimap $f : B \rightarrow C$ induces a functor $\vec{\pi}_1(f) : \vec{\pi}_1(B) \rightarrow \vec{\pi}_1(C)$.

Lemma 3.4 Given dihomotopic dimaps $f \simeq g : B \rightarrow C \in \mathbf{A} \downarrow \mathbf{Pospace}$ and a dipath $\gamma : \vec{I} \rightarrow B$ such that $\gamma(0) = \iota_B(a)$ and $\gamma(1) = \iota_B(b)$ where $a, b \in A$ then $f \circ \gamma \simeq g \circ \gamma$ are dihomotopic dipaths.

Proof. Since $f \simeq g$ there is a chain of dihomotopies $f \xrightarrow{H_1} f_1 \xleftarrow{H_2} f_2 \xrightarrow{H_3} \dots \xleftarrow{H_n} f_n \xrightarrow{H_{n+1}} g$. For $1 \leq i \leq n+1$, let $H'_i = H_i \circ (\gamma \times \vec{I})$.

$$\begin{array}{ccc} \vec{I} \times \vec{I} & & \\ \gamma \times \vec{I} \downarrow & \searrow^{H'_i} & \\ B \times \vec{I} & \xrightarrow{H_i} & C \end{array}$$

Then $f \circ \gamma \xrightarrow{H'_1} f_1 \circ \gamma \xleftarrow{H'_2} f_2 \circ \gamma \xrightarrow{H'_3} \dots \xleftarrow{H'_n} f_n \circ \gamma \xrightarrow{H'_{n+1}} g \circ \gamma$ is a chain of dihomotopies from the dipath $f \circ \gamma$ to the dipath $g \circ \gamma$. \square

⁵ This differs from the definition of fundamental category in [5] where the dihomotopy classes of dimaps use I and not \vec{I} .

Proposition 3.5 *If $f : B \rightarrow C \in \mathbf{A} \downarrow \mathbf{Pospace}$ is a dihomotopy equivalence then for all $a, b \in A$ the induced set map $\vec{\pi}_1(f)(a, b) : \vec{\pi}_1(B)(\iota_B(a), \iota_B(b)) \rightarrow \vec{\pi}_1(C)(\iota_C(a), \iota_C(b))$ is a bijection.*

Proof. By definition there is a dimap $g : C \rightarrow B$ such that $\vec{g} \circ f \simeq \text{Id}_B$ and $f \circ g \simeq \text{Id}_C$. So by Lemma 3.4, for any $a, b \in A$, any dipath $\gamma : \vec{I} \rightarrow B$ such that $\gamma(0) = \iota_B(a)$ and $\gamma(1) = \iota_B(b)$ and any dipath $\gamma' : \vec{I} \rightarrow C$ such that $\gamma(0) = \iota_C(a)$ and $\gamma(1) = \iota_C(b)$, $g \circ f \circ \gamma \simeq \gamma$ and $f \circ g \circ \gamma' \simeq \gamma'$. Hence $\vec{\pi}_1(g)(a, b)$ is an inverse for $\vec{\pi}_1(f)(a, b)$. \square

Example 3.6 Let $A = S^0 = \{a, b\}$ and choose any points $x, y \in \vec{I} \times \vec{I}$ such that $x \not\leq y$ and $y \not\leq x$. Then the sets $\vec{\pi}_1(\vec{I} \times \vec{I})(x, y)$ and $\vec{\pi}_1(\vec{I} \times \vec{I})(y, x)$ are empty. However for any dimap $f : \vec{I} \times \vec{I} \rightarrow \vec{I}$ (see Figure 2), either $f(x) \leq f(y)$ or $f(y) \leq f(x)$ since I is totally ordered. Furthermore one of $\vec{\pi}_1(\vec{I})(f(x), f(y))$ and $\vec{\pi}_1(\vec{I})(f(y), f(x))$ is nonempty. So in the context of $\iota_{\vec{I} \times \vec{I}}(a) = x$ and $\iota_{\vec{I} \times \vec{I}}(b) = y$, $\vec{I} \times \vec{I}$ is not dihomotopy equivalent to \vec{I} since there can be no dihomotopy equivalence $f : \vec{I} \times \vec{I} \rightarrow \vec{I}$ such that $\vec{\pi}_1(f)(a, b)$ is an isomorphism.

Example 3.7 Let \vec{X} be the space defined earlier (see Figure 1). In the context of its four endpoints $(0)_1, (0)_2, (1)_1$, and $(1)_2$, \vec{X} is not dihomotopy equivalent to \vec{I} (taking any four not necessarily distinct points as the context for \vec{I}). Indeed, there are no dipaths from $(0)_1$ to $(0)_2$ and vice versa (similarly for $(1)_1$ and $(1)_2$), whereas the same is not true for the corresponding points in \vec{I} .

4 Finding simpler models using context

In this section we look at two two-dimensional pospaces with a given context. We show how each of them can be replaced by an equivalent one-dimensional pospace by constructing explicit directed homotopy equivalences.

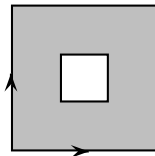


Fig. 4. $\vec{I} \times \vec{I}$ with a square removed

Example 4.1 In this example we show that in the context of the points $(0, 0)$ and $(1, 1)$, $\vec{I} \times \vec{I}$ with a square removed from its interior is dihomotopy equivalent to its boundary.

Let $A = S^0 = \{a, b\}$. Let B be the sub-pospace of $\vec{I} \times \vec{I}$ in Figure 4 given by $\vec{I} \times \vec{I} -]\frac{1}{3}, \frac{2}{3}[\times]\frac{1}{3}, \frac{2}{3}[$. Let $\iota_B(a) = (0, 0)$ and let $\iota_B(b) = (1, 1)$. Let C be the boundary of $\vec{I} \times \vec{I}$ with $\iota_C(a) = (0, 0)$ and $\iota_C(b) = (1, 1)$.

Intuitively we will contract B to C in two stages. First we will expand the missing square $(\frac{1}{3}, \frac{1}{3}) \times (\frac{1}{3}, \frac{1}{3})$ to $(\frac{1}{3}, 1) \times (\frac{1}{3}, 1)$ and then to $(0, 1) \times (0, 1)$. The first

will be done by a map h which we define below and the composite of the two will yield the desired dihomotopy equivalence f .

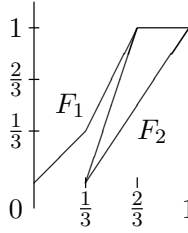


Fig. 5. The graphs of F_1 , F_2 , and $F_2 \circ F_1$.

Let $F_1 : [0, 1] \rightarrow [0, 1]$ be given by the mapping

$$x \mapsto \begin{cases} x & \text{if } x < \frac{1}{3} \\ 2x - \frac{1}{3} & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ 1 & \text{if } x > \frac{2}{3} \end{cases}$$

Let $F_2 : [0, 1] \rightarrow [0, 1]$ be given by the mapping

$$x \mapsto \begin{cases} 0 & \text{if } x < \frac{1}{3} \\ \frac{3}{2}x - \frac{1}{2} & \text{if } \frac{1}{3} \leq x \leq 1 \end{cases}$$

See Figure 5 for graphs of F_1 , F_2 , and $F_2 \circ F_1$.

Let $f : B \rightarrow C$ and $g : C \rightarrow B$ be given by $f(x, y) = (F_2 \circ F_1(x), F_2 \circ F_1(y))$ and $g(x, y) = (x, y)$. Also let $h : B \rightarrow B$ be given by $h(x, y) = (F_1(x), F_1(y))$. Since $F_2 \circ F_1$, Id_I , and F_1 are increasing maps, f , g , and h are dimaps.

We will now give explicit dihomotopies $\text{rel } A$ showing that $g \circ f \simeq \text{Id}_B \text{ rel } A$ and $f \circ g \simeq \text{Id}_C \text{ rel } A$. Let H_1 and H_2 be linear interpolations between Id_B and h and between $g \circ f$ and h . That is,

$$\begin{aligned} H_1(x, y, t) &= (1 - t)(x, y) + t(F_1(x), F_1(y)), \text{ and} \\ H_2(x, y, t) &= (1 - t)(F_2 \circ F_1(x), F_2 \circ F_1(y)) + t(F_1(x), F_1(y)). \end{aligned}$$

Note that F_1 and F_2 fix 0 and 1, so H_0 and H_1 fix the marked points $(0, 0)$ and $(1, 1)$. By Lemma 2.6, $H_1 : \text{Id}_B \xrightarrow{\simeq} h$ is a dihomotopy $\text{rel } A$. H_2 is a dimap since $F_2 \circ F_1(x) \leq F_1(x)$ for all $x \in I$, so $h \leq g \circ f$. Thus by Lemma 2.6, $H_2 : g \circ f \xrightarrow{\simeq} h$ is a dihomotopy $\text{rel } A$. Therefore $g \circ f \simeq \text{Id}_B \text{ rel } A$ as claimed. Furthermore since C is a sub-pospace of B and $f \circ g = f = g \circ f$, the above dihomotopies restrict to C showing that $f \circ g \simeq \text{Id}_C \text{ rel } A$.

Example 4.2 The Swiss flag.

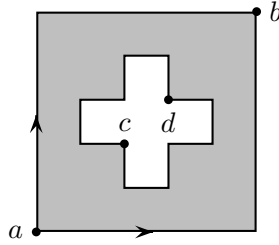


Fig. 6. The Swiss flag with labeled points $\{a, b, c, d\}$

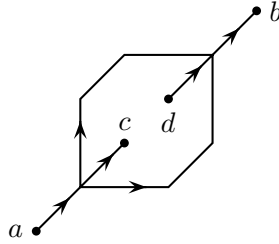


Fig. 7. A sub-pospace of the Swiss flag with the same labeled points $\{a, b, c, d\}$

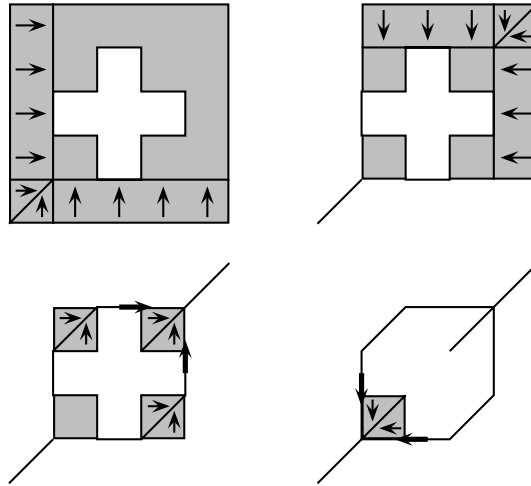


Fig. 8. An intuitive sketch of the dihomotopy equivalence between the Swiss flag and its sub-pospace

In this example we give an explicit dihomotopy between the famous Swiss flag pospace in Figure 6 and the one-dimensional sub-pospace in Figure 7 in the context of four points.

Let A be the discrete pospace $\{a, b, c, d\}$. Let B be the sub-pospace of $\vec{I} \times \vec{I}$ given in Figure 6 with the (open) cross removed and $\iota_B(a) = (0, 0)$, $\iota_B(b) = (1, 1)$, $\iota_B(c) = (\frac{2}{5}, \frac{2}{5})$, and $\iota_B(d) = (\frac{3}{5}, \frac{3}{5})$. Let C be the subspace of B given in Figure 7 with the same marked points.

Intuitively we will contract B to C by applying four maps which are described in Figure 8.

Let $g : C \rightarrow B$ be the dimap given by $g(x, y) = (x, y)$. Let $f : B \rightarrow C$ be the dimap given by $f(x, y) = f_4 \circ f_3 \circ f_2 \circ f_1(x, y)$ where f_1 , f_2 , f_3 , and f_4 are defined in (1) below. From the sketches of f_1 , f_2 , f_3 , and f_4 in Figure 8, one can see that

they are dimaps. As in the previous example we will give a chain of dihomotopies rel A to show that $\text{Id}_B \simeq g \circ f$. Since C is a subspace of B and $g \circ f = f = f \circ g$ this will restrict to a chain of dihomotopies rel A which show that $\text{Id}_C \simeq f \circ g$. As a result we will have that $B \simeq C$.

$$\begin{aligned}
 f_1(x, y) &= \begin{cases} (\max(x, y), \max(x, y)) & \text{if } 0 \leq x \leq \frac{1}{5}, 0 \leq y \leq \frac{1}{5} \\ (\frac{1}{5}, y) & \text{if } 0 \leq x \leq \frac{1}{5}, \frac{1}{5} < y \\ (x, \frac{1}{5}) & \text{if } 0 \leq y \leq \frac{1}{5}, \frac{1}{5} < x \\ (x, y) & \text{otherwise} \end{cases} \\
 f_2(x, y) &= \begin{cases} (\min(x, y), \min(x, y)) & \text{if } \frac{4}{5} \leq x \leq 1, \frac{4}{5} \leq y \leq 1 \\ (\frac{4}{5}, y) & \text{if } \frac{4}{5} \leq x \leq 1, y < \frac{4}{5} \\ (x, \frac{4}{5}) & \text{if } \frac{4}{5} \leq y \leq 1, x < \frac{4}{5} \\ (x, y) & \text{otherwise} \end{cases} \\
 f_3(x, y) &= \begin{cases} (\max(x, y - \frac{2}{5}), \max(x + \frac{2}{5}, y)) & \text{if } \frac{1}{5} \leq x \leq \frac{2}{5}, \frac{3}{5} \leq y \leq \frac{4}{5} \\ (\max(x, y + \frac{2}{5}), \max(x - \frac{2}{5}, y)) & \text{if } \frac{1}{5} \leq y \leq \frac{2}{5}, \frac{3}{5} \leq x \leq \frac{4}{5} \\ (\max(x, y), \max(x, y)) & \text{if } \frac{3}{5} \leq x \leq \frac{4}{5}, \frac{3}{5} \leq y \leq \frac{4}{5} \\ (\frac{2}{5} + 2(x - \frac{2}{5}), y) & \text{if } \frac{2}{5} \leq x \leq \frac{3}{5}, y = \frac{4}{5} \\ (x, \frac{2}{5} + 2(y - \frac{2}{5})) & \text{if } \frac{2}{5} \leq y \leq \frac{3}{5}, x = \frac{4}{5} \\ (x, y) & \text{otherwise} \end{cases} \\
 f_4(x, y) &= \begin{cases} (\min(x, y), \min(x, y)) & \text{if } \frac{1}{5} \leq x \leq \frac{2}{5}, \frac{1}{5} \leq y \leq \frac{2}{5} \\ (\frac{3}{5} - 2(\frac{3}{5} - x), y) & \text{if } \frac{2}{5} \leq x \leq \frac{3}{5}, y = \frac{1}{5} \\ (x, \frac{3}{5} - 2(\frac{3}{5} - y)) & \text{if } \frac{2}{5} \leq y \leq \frac{3}{5}, x = \frac{1}{5} \\ (x, y) & \text{otherwise} \end{cases}
 \end{aligned} \tag{1}$$

Let H_1, H_2, H_3 , and H_4 be the linear interpolations (see Definition 2.4) between Id_B and f_1 , $f_2 \circ f_1$ and f_1 , $f_2 \circ f_1$ and $f_3 \circ f_2 \circ f_1$, and f and $f_3 \circ f_2 \circ f_1$. Since the f_i fix the labeled points, so do the H_i . Furthermore, since f_1, f_2, f_3 , and f_4 are dimaps, f_1 and f_3 are increasing and f_2 and f_4 are decreasing, by Lemma 2.6, the H_i form a chain of dihomotopies

$$\text{Id}_B \xrightarrow{H_1} f_1 \xleftarrow{H_2} f_2 \circ f_1 \xrightarrow{H_3} f_3 \circ f_2 \circ f_1 \xleftarrow{H_4} f = g \circ f.$$

Therefore $\text{Id}_B \simeq g \circ f$. Restricting to C gives a chain of dihomotopies showing

$\text{Id}_C \simeq f = f \circ g$. Hence B is dihomotopy equivalent to $C \text{ rel } \{a, b, c, d\}$.

5 Pushouts of dihomotopy equivalences

In this section we elaborate on the statement made in the introduction that dihomotopy equivalences should be preserved by ‘pastings’. In fact we discuss the construction of a homotopy theory for concurrency. In order that we do not lose focus from the main ideas of this paper, we will defer the details of the definitions and constructions of this section to the appendix.

An excellent framework for a homotopy theory on a category is given by a *model structure* on the category [12]. A category with a model structure and all small limits and colimits is called a *model category*. A model structure has three special classes of morphisms: *fibrations*, *cofibrations*, and *weak equivalences* which satisfy certain axioms (see Appendix A for the full definition).

The category **Pospace** has all small limits and colimits. However it is too restrictive to model many concurrent systems (for example pospaces cannot contain loops). Though all of our examples are in **Pospace** a better framework for concurrency is the category **LoPospc** of *local pospaces*. A *local pospace* is a topological space such that each point has a neighborhood which is a pospace and that these local orders are compatible (for a precise definition see Appendix B).



Fig. 9. The local pospace \vec{S}^1

Example 5.1 An example of a local pospace is the directed circle \vec{S}^1 in Figure 9 obtained by identifying the endpoints of \vec{I} . While \vec{S}^1 does not have a transitive, anti-symmetric order, locally it has the structure of the pospace \vec{I} .

Unfortunately, unlike **Pospace**, **LoPospc** does not contain all small colimits. However there is a formal method of enlarging a category to one with all small limits and colimits.⁶ Furthermore this larger category has a canonical model structure! [2] For details on how this theory can be applied to **LoPospc** see the appendix and [1]. In the appendix we give a more precise version of the following theorem (Theorem B.4) which is proved in [1].

Theorem 5.2 *Let $\mathbf{C} = \mathbf{LoPospc}$. Then \mathbf{C} is a subcategory of a model category \mathbf{UC} . The morphisms in \mathbf{C} that are cofibrations are the monomorphisms and the morphisms in \mathbf{C} that are weak equivalences are the isomorphisms.*

From the point of view of just \mathbf{C} , this model structure is almost trivial. However one can *localize* \mathbf{UC} with respect to a set M of morphisms in \mathbf{C} to obtain a new category \mathbf{UC}/M . \mathbf{UC}/M has the same objects and cofibrations as \mathbf{UC} but the morphisms in M are now weak equivalences [2]. The problem is to choose a good

⁶ Again more details are provided in the appendix (one passes to the category of simplicial presheaves [2]).

set of morphisms M . For example, we can take M to be the set of dihomotopy equivalences in \mathbf{C} .

One of the key properties of \mathbf{UC} and \mathbf{UC}/\mathbf{M} is that they are *left proper*. That is, the pushout of a weak equivalence over a cofibration is a weak equivalence.

$$\begin{array}{ccc} G & \xrightarrow{\sim} & C \\ j \downarrow & & \downarrow \\ D & \xrightarrow{\sim} & E \end{array}$$

In particular in \mathbf{UC}/\mathbf{M} if $f \in M$ then g is a weak equivalence.

Example 5.3 Recall the dihomotopy equivalence $f : \vec{I} \times \vec{I} \rightarrow \vec{I}$ of Example 2.12. Also recall the inclusions of $\vec{I} \times \vec{I}$ and \vec{I} into B and C (see Figure 3) given in Example 1.2 where attachments are made at the points $x, y \in \vec{I} \times \vec{I}$ and $x', y' \in \vec{I}$ (see Figure 2). We have the following pushout diagram.

$$\begin{array}{ccc} \vec{I} \times \vec{I} & \xrightarrow{\sim} & \vec{I} \\ j \downarrow & & \downarrow \\ B & \xrightarrow{g} & C \end{array}$$

Since the inclusion j is a cofibration, we get a weak equivalence between B and C . However as discussed in Example 1.2, B should not be equivalent to C .

The solution to this problem is to work with $\mathbf{A} \downarrow \mathbf{LoPospc}$ instead of $\mathbf{LoPospc}$ where the choice of context $A \in \mathbf{Ob} \mathbf{LoPospc}$ depends on the pushouts that one would like to consider.

In the example above the right context is clearly the points $x, y \in \vec{I} \times \vec{I}$ and $x', y' \in \vec{I}$. So $A = \{a, b\}$, $\iota_{\vec{I} \times \vec{I}}(a) = x$, $\iota_{\vec{I} \times \vec{I}}(b) = y$, $\iota_{\vec{I}}(a) = x'$, and $\iota_{\vec{I}}(b) = y'$. As discussed in Example 3.6 the map f is not a dihomotopy equivalence rel A . So we are not forced to conclude that there is a weak equivalence between B and C .

In the following two examples we examine the ‘pastings’ of two copies of $\vec{I} \times \vec{I}$ with a square removed. We show how choosing the right context allows us to find a one-dimensional sub-pospace which is dihomotopy equivalent to the pushout.

Unlike the previous section, we will not write out the explicit dihomotopy equivalences in these two examples.

Example 5.4 Let A be the discrete space $\{a, b, c\}$. Let B be the subspace of $\vec{I} \times \vec{I}$ in Figure 10 with the square $\{(x, y) \mid \frac{1}{5} < x < \frac{2}{5}, \frac{2}{5} < y < \frac{3}{5}\}$ removed. Let $\iota_B(a) = (0, 0)$, $\iota_B(b) = (\frac{1}{2}, 0)$, and $\iota_B(c) = (1, 0)$.

Let C be the subspace of $\vec{I} \times \vec{I}$ in Figure 11 with the square $\{(x, y) \mid \frac{3}{5} < x < \frac{4}{5}, \frac{2}{5} < y < \frac{3}{5}\}$ removed. Let $\iota_C(a) = (0, 1)$, $\iota_C(b) = (\frac{1}{2}, 1)$, and $\iota_C(c) = (1, 1)$.

Let B' be the subspace of $\vec{I} \times \vec{I}$ in Figure 10 with the rectangle $]0, \frac{1}{2}[\times]0, 1[$ removed and the same marked points. Then there is a dihomotopy equivalence $f : B \xrightarrow{\sim} B'$ rel A . One can construct the required dihomotopies by stretching the

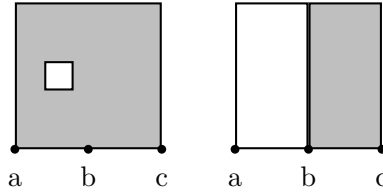


Fig. 10. The spaces B and B' , which are subspaces of $\vec{T} \times \vec{T}$ with a rectangle removed and labeled points $\{a, b, c\}$

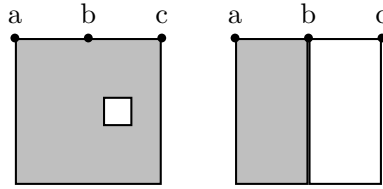


Fig. 11. The spaces C and C' , which are subspaces of $\vec{T} \times \vec{T}$ with a square removed and labeled points $\{a, b, c\}$

region $\frac{2}{5} \leq y \leq \frac{3}{5}$ first to $y = 1$ and then to $y = 0$. Next one stretches the region $\frac{1}{5} \leq x \leq \frac{2}{5}$ first to $x = \frac{1}{2}$ and then to $x = 0$. All this is done while leaving the three marked points fixed.

Similarly there is a dihomotopy equivalence $g : C \xrightarrow{\sim} C'$ rel A where C' is the subspace of $\vec{T} \times \vec{T}$ in Figure 11 with the rectangle $\{(x, y) | \frac{1}{2} < x < 1, 0 < y < 1\}$ removed.

Let D be the space obtained by attaching B along its bottom edge to the top edge of C . Notice that $D \in \mathbf{Ob} \mathbf{A} \downarrow \mathbf{Pospace}$ and the inclusions $i : B \rightarrow D$ and $j : C \rightarrow D$ are dimaps in $\mathbf{A} \downarrow \mathbf{Pospace}$.

Now take the following pushout.

$$\begin{array}{ccc} B & \xrightarrow{\sim} & B' \\ i \downarrow & f & \downarrow \\ D & \xrightarrow{\sim} & E \end{array}$$

Then E is the pospace obtained by attaching the bottom edge of B' to the top edge of C . Since C includes into E we can take the following pushout.

$$\begin{array}{ccc} C & \xrightarrow{\sim} & C' \\ i \downarrow & g & \downarrow \\ E & \xrightarrow{\sim} & F \end{array}$$

Now F is the pospace⁷ in Figure 12 obtained by attaching the bottom edge of B' to the top edge of C' .

⁷ Being precise, if we consider the pushout F to be a subspace of $I \times I$ then the points $[0, \frac{1}{5}] \times \{\frac{1}{2}\}$ are identified as are the points $[\frac{2}{5}, \frac{3}{5}] \times \{\frac{1}{2}\}$ and the points $[\frac{4}{5}, 1] \times \{\frac{1}{2}\}$. However this pospace is dihomotopy equivalent to the pospace obtained by attaching the bottom edge of B' to the top edge of C' .

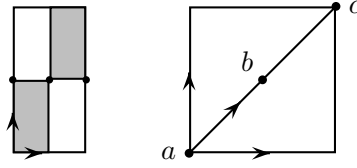


Fig. 12. The pospaces F and G

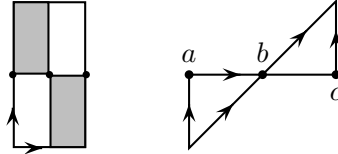


Fig. 13. The pospaces F' and G'

Finally F is dihomotopy equivalent rel A to the space G in Figure 12. Consider F and G as sub-pospaces of $\vec{I} \times \vec{I}$. The dihomotopy is obtained by first collapsing the square $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ using $(x, y) \mapsto (\max(x, y), \max(x, y))$, and then collapsing the square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ using $(x, y) \mapsto (\min(x, y), \min(x, y))$

Thus in the context of A , D is equivalent to G .

Example 5.5 Let A, B, C, B' and C' be as in the previous example, except that the marked points on B and B' are taken to be on the top edge, and the marked points on C and C' are taken to be on the bottom edge. Let D' be the space obtained by attaching C along its bottom edge to the top edge of B .

Then as in the previous example D' is dihomotopy equivalent to F' where F' is the pospace in Figure 13 obtained by attaching the bottom edge of C' to the top edge of B' .

Finally F' is dihomotopy equivalent rel A to the space G' in Figure 13. Consider F' and G' as sub-pospaces of $\vec{I} \times \vec{I}$. The dihomotopy is obtained by collapsing the regions $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ using $(x, y) \mapsto (x, \frac{1}{2})$, and then collapsing the square $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ using $(x, y) \mapsto (x, \frac{1}{2})$.

Thus in the context of A , D' is equivalent to G' .

Example 5.6 Finally we give an example which requires a non-discrete context. Let $X = \vec{I} \times \vec{I}$. We will show that if we want to use X to construct a certain space Z then there is no appropriate finite context.

Let I be the unit interval $[0, 1]$ together with the trivial partial order given by $x \leq_I y$ if and only if $x = y$.

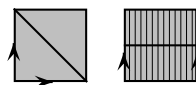


Fig. 14. X and Y with the images of I marked

Let $\varphi : I \rightarrow X$ be the inclusion of the anti-diagonal, given by $t \mapsto (t, 1 - t)$ (see Figure 14). Let $Y = I \times \vec{I}$ and let $\psi : I \rightarrow Y$ be the inclusion of the central line, given by $t \mapsto (t, \frac{1}{2})$ (see Figure 14). Define the pospace Z obtained by gluing X and

Y together along the images of I . That is, Z is the following pushout.

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & X \\ \psi \downarrow & & \downarrow \iota_X \\ Y & \xrightarrow{\iota_Y} & Z \end{array}$$

We claim that if we want to consider this pushout then there is no appropriate finite context.

For $\alpha \in I$ let $p_\alpha := \iota_X(\varphi(\alpha)) = \iota_Y(\alpha, \frac{1}{2})$, $p_\alpha^0 = \iota_Y(\alpha, 0)$ and $p_\alpha^1 = \iota_Y(\alpha, 1)$. Notice that for $s \neq t \in I$ there does not exist a dipath in Z from p_s^0 to p_t^1 .

Now let A be some context and fix $\iota_I : A \rightarrow I$ which determines $\iota_X : A \rightarrow X$ and $\iota_Y : A \rightarrow Y$. Let $f : X \rightarrow X'$ be some dihomotopy equivalence rel A . Let Z' and g be defined by the following pushout.

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \iota_X \downarrow & & \downarrow \\ Z & \xrightarrow{g} & Z' \end{array}$$

Assume there exists $s \neq t \in I$ such that $f(\varphi(s)) = f(\varphi(t))$. We claim that there is a dipath from $g(p_s^0)$ to $g(p_t^1)$. In Z there is a dipath from p_s^0 to p_s and a dipath from p_t to p_t^1 . The concatenation of the images of these paths under g gives the desired dipath in Z' . But this contradicts Proposition 3.5.

Therefore there should not have been an equivalence f such that $f(\varphi(s)) = f(\varphi(t))$ for some $s \neq t \in I$. We can prevent this difficulty if we use the context $A = I$ together with $\iota_I = \text{Id}_I$.

On the other hand with any finite context A , we claim that there is always such a dihomotopy equivalence f . Assume that A is a finite context and fix $\iota_I : A \rightarrow I$. Since A is finite there is some interval $[a, b] \subset I \setminus \iota_I(A)$. Let $f : X \rightarrow X$ be the dihomotopy equivalence given by the concatenation of the following two maps. First collapse the region $[a, b] \times I$ to the right. Then collapse the region $I \times [1 - b, 1 - a]$ upwards. Then f is a dihomotopy equivalence rel A but $f(\varphi(a)) = f(\varphi(b))$.

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References

- [1] Bubenik, P. and K. Worytkiewicz, *A model category structure for local po-spaces*, arXiv:math.AT/0506352 (2005).
URL <http://arxiv.org/abs/math.AT/0506352>
- [2] Dugger, D., *Universal homotopy theories*, Adv. Math. **164** (2001), pp. 144–176.

- [3] Fajstrup, L., *Discovering spaces*, Homology Homotopy Appl. **5** (2003), pp. 1–17 (electronic), Algebraic topological methods in computer science (Stanford, CA, 2001).
- [4] Fajstrup, L., E. Goubault and M. Raussen, *Algebraic topology and concurrency*, to appear in Theoretical Computer Science (1999), also preprint R-99-2008, Dept. of Mathematical Sciences, Aalborg University, Aalborg, Denmark.
- [5] Fajstrup, L., M. Raussen, E. Goubault and E. Haucourt, *Components of the fundamental category*, Appl. Categ. Structures **12** (2004), pp. 81–108.
- [6] Gaucher, P., *A model category for the homotopy theory of concurrency*, Homology Homotopy Appl. **5** (2003), pp. 549–599.
- [7] Goubault, E., “The geometry of concurrency,” Ph.D. thesis, Ecole Normale Supérieure (1995).
- [8] Goubault, E., *Some geometric perspectives in concurrency theory*, Homology Homotopy Appl. **5** (2003), pp. 95–136 (electronic), Algebraic topological methods in computer science (Stanford, CA, 2001).
- [9] Grandis, M., *Directed homotopy theory. I*, Cah. Topol. Géom. Différ. Catég. **44** (2003), pp. 281–316.
- [10] Grandis, M., *Directed homotopy theory. II. Homotopy constructs*, Theory Appl. Categ. **10** (2002), pp. No. 14, 369–391 (electronic).
- [11] Hirschhorn, P. S., “Model categories and their localizations,” Mathematical Surveys and Monographs **99**, American Mathematical Society, Providence, RI, 2003, xvi+457 pp.
- [12] Hovey, M., “Model categories,” Mathematical Surveys and Monographs **63**, American Mathematical Society, Providence, RI, 1999, xii+209 pp.
- [13] Raussen, M., *State spaces and dipaths up to dihomotopy*, Homology Homotopy Appl. **5** (2003), pp. 257–280 (electronic), Algebraic topological methods in computer science (Stanford, CA, 2001).

A Model Categories

In this section we define model categories, and show how a given small category can be embedded into a *universal model category*. For more details see [2,1].

Definition A.1 A *model category* is a category \mathbf{C} with three distinguished classes of morphisms: weak equivalences, cofibrations, and fibrations satisfying the following conditions:

- (i) \mathbf{C} contains all small limits and colimits.
- (ii) If there exist morphisms f , g and $g \circ f$ and two of them are weak equivalences then so is the third.
- (iii) Weak equivalences, cofibrations, and fibrations are closed under retracts.
- (iv) Given any commutative diagram

$$\begin{array}{ccc} A & & > X \\ i \downarrow & & \downarrow p \\ B & & > Y \end{array}$$

such that i is a cofibration and p is a fibration, then if either i or p is also a weak equivalence then there exists a map $B \rightarrow X$ making the diagram commute.

- (v) Any map may be factored as a cofibration followed by a fibration which is a weak equivalence, and as a cofibration which is a weak equivalence followed by a fibration.

Next we define the category of simplicial presheaves.

Definition A.2 • The simplicial category Δ is the category whose objects are $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and whose morphisms are maps $f : [n] \rightarrow [k]$ such that $x \leq y$ implies that $f(x) \leq f(y)$.

- The category of simplicial sets \mathbf{sSet} is the category $\mathbf{Set}^{\Delta^{\text{op}}}$ whose objects are contravariant functors from Δ to the category of sets \mathbf{Set} and whose morphisms are natural transformations.
- Let \mathbf{C} be a small category. Then $\mathbf{sPre}(\mathbf{C})$ is the category $\mathbf{sSet}^{\mathbf{C}^{\text{op}}}$ whose objects are the contravariant functors from \mathbf{C} to \mathbf{sSet} and whose morphisms are natural transformations.

Remark A.3 An important fact is that there is an embedding $\mathbf{C} \rightarrow \mathbf{sPre}(\mathbf{C})$.

The category \mathbf{sSet} has a model structure in which the cofibrations are the monomorphisms and the weak equivalences are the morphisms f such that $|f|$ the geometric realization of f is a weak equivalence in the category of topological spaces (that is, it induces isomorphisms between homotopy groups). For more details see [12].

The category of simplicial presheaves has a canonical model structure, called the *cofibrant model structure*, where the weak equivalences and the cofibrations are defined objectwise. That is, a morphism f in $\mathbf{sPre}(\mathbf{C})$ is a weak equivalence or cofibration if and only if for each $X \in \text{Ob } \mathbf{C}$ the morphism $f(X)$ is a weak equivalence or cofibration in \mathbf{sSet} .

Now one can localize this model category [11] with respect some set of morphisms M to get a new model category $\mathbf{sPre}(\mathbf{C})/M$. This model category has the same objects, but in addition to the previous weak equivalences, the morphisms in M are now weak equivalences. For example if $\mathbf{C} = \mathbf{LoPospc}$ then one could localize with respect to all dihomotopy equivalences (it makes sense to say this because of the embedding of \mathbf{C} in $\mathbf{sPre}(\mathbf{C})$).

B Local po-spaces

In this section we give a precise definition of the category $\mathbf{LoPospc}$ of local pospaces and use it to give a more precise version of Theorem 5.2. Local pospaces are defined in [4,1]. Here we follow [1].

Definition B.1 • Given a topological space M , an *order atlas* on M is an open cover⁸ $U = \{U_i\}$ indexed by a set I such that each U_i is a pospace and that the orders are compatible. That is, given $x, y \in U_i \cap U_j$, $x \leq_i y$ if and only if $x \leq_j y$.

- Let $U = \{U_i\}$ and $V = \{V_j\}$ be two order atlases. Then V is said to be a *refinement* of U if for any U_i and any $x \in U_i$ there exists a V_j containing x which is a sub-pospace of U_i .

⁸ That is, each U_i is an open subset of M , and $M = \cup_{i \in I} U_i$.

- Two order atlases are said to be *equivalent* if they have a common refinement. One can check that this defines an equivalence relation.
- Define a *local pospace* to be a topological space together with an equivalence class of order atlases.
- Define a *dimap of local pospaces* $f : (M, \bar{U}) \rightarrow (N, \bar{V})$ to be a continuous map $f : M \rightarrow N$ such that for any choice of $V = \{V_j\} \in \bar{V}$ there is some choice of $U = \{U_i\} \in \bar{U}$ such that for all i, j the partial map $f : U_i \rightarrow V_j$ is a dimap of pospaces.

Definition B.2 Define **LoPospc** to be the category whose objects are local pospaces whose underlying topological spaces are subsets of \mathbb{R}^n for some n ,⁹ and whose morphisms are dimaps between local pospaces.

Remark B.3 Notice that we have restricted the class of local pospaces in our category. This is done precisely so that the resulting category **LoPospc** is a small category, which is used to apply the machinery of Appendix A. For the purposes of concurrency, this does not seem to be a significant limitation. Furthermore, it may be possible that any local pospace can be ‘found’ in **sPre(LoPospc)**.

Nevertheless, a consequence of this, is that the category **Pospace** in Definition 2.2 is not a subcategory of **LoPospc**. Of course one could define a new category **Pospace'** whose objects are those pospaces whose underlying topological spaces are subsets of \mathbb{R}^n for some n . Then **Pospace'** is a subcategory of **LoPospc**. All of our examples are in **Pospace'**.

We can now give a more precise version of Theorem 5.2.

Theorem B.4 ([1]) *There exists a model structure on **sPre(LoPospc)** such that the cofibrations are the monomorphisms. Furthermore the morphisms in **LoPospc** which are weak equivalences in **sPre(LoPospc)** are just the isomorphisms.*

⁹ The local partial order need not be the one inherited from the usual partial order on \mathbb{R}^n .